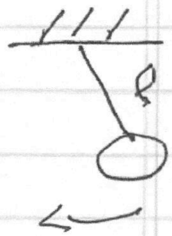
# Adiabatic Invariants

## Adiabatic Invariants

→ 

$l = l(t)$

s/t $\boxed{\dfrac{\dot{l}}{l} < \sqrt{g/l}}$

→ 2 time scale

→ 'adiabatic' variation of parameter.

→ How describe?

→ Now $\ddot{\Theta} + \dfrac{g}{l(t)}\Theta = 0$

essence is oscillator with slowly varying parameter

$l(t) = l(\epsilon t)$
↓
slow

$\ddot{\Theta} + \dfrac{g}{l(\epsilon t)}\Theta = 0$

$\epsilon t = \tau$

$\Rightarrow dt = \dfrac{1}{\epsilon} d\tau$

$d/dt = \epsilon \dfrac{d}{d\tau}$

$\ddot{\Theta} + \dfrac{g}{\epsilon^2 l(\tau)}\Theta = 0 \Rightarrow \boxed{\ddot{\Theta} + \dfrac{\omega(\tau)^2}{\epsilon^2}\Theta = 0}$

generically points toward WKB.

i.e. generic:

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

$\Rightarrow \tau = \epsilon t$

$$\frac{d^2x}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2}\, x = 0$$

how handle variable frequency ?

→ A different look at adiabatic theory . . .

One might forego canonical formalism, and simply investigate an oscillator with slowly varying frequency

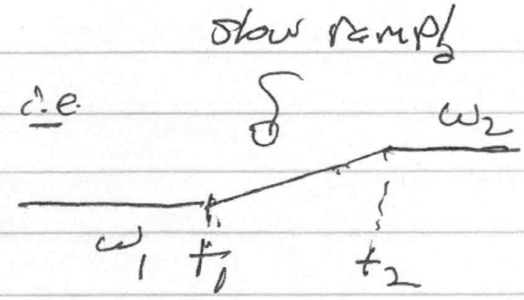
i.e.

$$\ddot{x} + \omega^2 x = 0 \quad \Rightarrow$$

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

slowly varying frequency

i.e.




i.e.

$$\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon\omega) \ll 1$$

⇒ expect, on basis of previous discussion,

$I \leftrightarrow$ adiabatic invariant

will show

i.e. if $a \equiv$ oscillator amplitude, then

$$I = E/\omega = \tfrac{1}{2} m\omega^2 a^2/\omega \simeq m\omega a^2$$

as const

→ action!

Need show action is ~ const, adiabatic invariant.

Now for slowly varying $\omega$, can solve by WKB ↓

Now $\epsilon t = T$

$$\frac{d^2 x}{dT^2} + \frac{\omega^2(T)}{\epsilon^2} x = 0$$

$$x(T) = a_0 e^{i\phi(T)/\epsilon}$$

where $\phi = \phi_0 + \epsilon \phi_1 + \ldots$

$$\frac{d}{dT}\left( a_0 \frac{i\dot{\phi}(T)}{\epsilon} e^{i\frac{\phi(T)}{\epsilon}} \right) + \frac{\omega(T)^2}{\epsilon^2} a_0 e^{i\phi(T)} = 0$$

$$\left( -\frac{\dot{\phi}^2}{\epsilon^2} + \frac{i\ddot{\phi}}{\epsilon} \right) a_0 e^{i\phi} + \frac{\omega(T)^2}{\epsilon^2} a_0 e^{i\phi} = 0$$

and then expanding $\phi$

$$\left( -\frac{(\dot{\phi}_0 + \epsilon\dot{\phi}_1)^2}{\epsilon^2} + \frac{i\ddot{\phi}_0(T)}{\epsilon} \right) + \frac{\omega(T)^2}{\epsilon^2} = 0$$

l.o. $-\dfrac{\dot{\phi}_0^2}{\epsilon^2} + \dfrac{\omega(T)^2}{\epsilon^2} = 0 \qquad O(1/\epsilon^2)$

so $\quad \dot{\phi}_0(T) = \omega(T)$

$$\phi_0(T) = \int d\tilde{T}\, \omega(\tilde{T})$$

1st Order: $\quad -2\frac{\dot{\phi}_0}{\epsilon}\dot{\phi}_1 + i\frac{\ddot{\phi}_0}{\epsilon} = 0 \qquad O(1/\epsilon)$

$$\dot{\phi}_1 = i\ddot{\phi}_0 / 2\dot{\phi}_0$$

$$= \frac{i}{2}\frac{d}{dt}\ln\left(\dot{\phi}_0(T)\right)$$

$$\phi_1 = \frac{i}{2}\ln\left(\dot{\phi}_0(T)\right)$$

$$= \frac{i}{2}\ln\left(\omega(T)\right)$$

so

$$X(T) = \overset{\text{amp}}{a_0}\, e^{i\phi(T)/\epsilon}$$

$$= a_0\, e^{i\int \frac{\omega}{\epsilon}(\tilde{T})\, d\tilde{T}}\, e^{i\frac{i}{2}\ln(\omega(T))}$$

$$= a_0\, e^{i\int \frac{\omega}{\epsilon}(\tilde{T})\, d\tilde{T}}\, e^{-\ln\omega(T)/2}$$

so

$$\boxed{X = a_0/\sqrt{\omega(T)}\; e^{i\int \frac{\omega}{\epsilon}(\tilde{T})\, d\tilde{T}}}$$

$$x(\tau) = \frac{a_0}{\sqrt{\omega(\tau)}} e^{i\int \frac{\omega(\tau)}{\epsilon} d\tau}$$

re-scaling $t = \tau/\epsilon$ $\quad d\tau = \epsilon dt$

$$\boxed{x(t) = \frac{a_0}{\sqrt{\omega(\epsilon t)}} e^{i\int \omega(\epsilon t)\, dt}}$$

WKB Soln.

And immediately observe:

$$\overline{\omega x^2} = \cancel{\omega} \frac{a_0^2}{2\cancel{\omega}} = \text{const}$$

$$\text{avg} = \int_0^{2\pi/\omega} \frac{dt}{T}$$

but $\quad \overline{\omega x^2} = \frac{\overline{\omega^2 x^2}}{\omega} \sim \frac{E}{\omega} \sim \text{Action}$

$\sim I$

N.B. $\rightarrow$ $I \sim E/\omega$ is invariant.
Action as 'adiabatic invariant', as $\omega(\epsilon t)$ evolves slowly.

- can 'discover' from WKB calculation
- Action is invariant, due to frequency modulation of amplitude!

Check: $$I = \frac{1}{2\pi}\oint p\,dq$$

$$= \frac{1}{2\pi}\oint p\,dx$$

$$= \frac{1}{2\pi}\oint m\dot{x}\,dx = \frac{1}{2\pi}\oint m\dot{x}^2\,dt$$

$$I = \frac{1}{2\pi}\oint_{\omega^{-1}} m\dot{x}^2\,dt$$

$$x(t) = \frac{a_0}{\sqrt{\omega}}\cos(\omega t + \delta)$$

$$\dot{x} = -a_0\sqrt{\omega}\,\sin(\omega t + \delta)$$

$\theta = \omega t$

$\cancel{d\theta} = \omega\,dt$

$\delta \to 0$

$$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\int d\theta\, a_0^2\,\frac{\omega^2 \sin^2\theta}{\omega}\,\frac{d\theta}{\omega}$$

$I \sim$ const. $\quad = a_0^2/2 \to$ real const.

The message:

- adiabatic invariants basically a consequence of WKB approximation, due time scale separation

- WKB could lead one to forming the theory of adiabatic invariants, even if one did not initially realize it.

- need retain WKB correction beyond pure eikonal for frequency modulation of amplitude $\Rightarrow$ essential.

→ Adiabatic Invariants [and Action - Angle Variables]

i) Adiabatic Invariants — Formal Approach (Bounded phase space ⟺ QPO)

→ Consider finite motion in 1D. Motion characterized by $\lambda$ parameter, such that:

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \frac{1}{T}$$

↳ period of motion

i.e. $\ell(t)$, $m$ $\qquad \frac{1}{\ell}\frac{d\ell}{dt} < \sqrt{g/\ell}$ $\qquad$ pull on string

thus, $\dot{E}$ will be "small / slow". (i.e. $H = H(\lambda(t), p, q)$

parametric dependence,

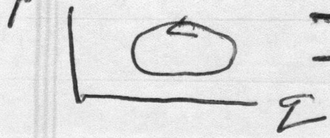
Now, $\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$

as $\lambda$ varies slowly compared to $\omega_0 = 1/T$, can average over $t$ on fast scales, i.e.

$$\frac{d\overline{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}} \simeq \overline{\frac{\partial H}{\partial \lambda}}\frac{d\lambda}{dt}$$

↳ avg. over motion $q, p$ → fast

break avg. on basis time scale separation



$-\omega$

$-\dot{\omega}/\omega$

[redacted]

where $\overline{A} = \frac{1}{T}\int^{\tau} dt\, A(t)$ → holding $E, \lambda$

average on fast timescale fixed!

$\Rightarrow$

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^{\tau} \frac{\partial H}{\partial \lambda}\, dt$$

Now,

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$dt = \int dq \big/ \partial H/\partial p$$

∴ can take $\int_0^{T} dt \to \oint \frac{dq}{\partial H/\partial p}$

$\oint \leftrightarrow$ complete circuit orbit

so finally,

avg

$$\frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\left\{\frac{\oint (\partial H/\partial \lambda)\, dq \big/ (\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}\right\}$$

$$\equiv \frac{d\lambda}{dt}\left\langle \frac{\partial H}{\partial \lambda}\right\rangle_{\tau}.$$

Now: – integrations must be performed for fixed, given value of $\lambda$ (i.e. $\dot{\lambda}/\lambda \ll \omega_0$)
– on such path (n.b. why "path" of interest!), $H = E$ and $p = p(q; E, \lambda)$

$\therefore \quad H(p, q, \lambda) = E$    {path for E const.

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{d\lambda} = 0$$

$$\Rightarrow \quad \frac{\partial H/\partial \lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$$

plug in previous

$$\therefore \quad \frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\,\frac{\oint -(\partial p/\partial \lambda)\,dq}{\oint dq\,\partial p/\partial E}$$

$(1/\partial H/\partial p = \partial p/\partial E)$    (fixed $\lambda$)

so, re-writing:

$$\frac{d\bar{E}}{dt}\oint dq\,\partial p/\partial E + \frac{d\lambda}{dt}\oint (\partial p/\partial \lambda)\,dq = 0$$

$$\Rightarrow \oint_{E,\lambda \text{ fixed}} dq \left\{ \frac{\partial p}{\partial E}\frac{dE}{dt} + \frac{\partial p}{\partial \lambda}\frac{d\lambda}{dt} \right\} = 0$$

$$\Rightarrow \boxed{\overline{dI/dt} = 0}$$

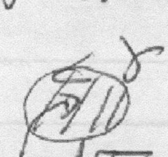
where $I = \oint_{E,\lambda \text{ fixed}} \frac{p\,dq}{2\pi}$ → {integral taken over path for fixed, given $E$, $\lambda$}

$\therefore$ → $\boxed{\begin{array}{l} I \text{ const. as } \lambda \text{ varies!} \\ \therefore\ I \text{ adiabatic invariant} \end{array}}$ (abbreviated action along approx. traj.)

→ in general (including higher dimensions)

$$I_C = \oint_\gamma \underline{p} \cdot d\underline{q} = \iint_\sigma d\underline{p} \wedge d\underline{q}$$ {Liouville Thm, again}

is Poincares relative integral invariant ($\gamma$ closed curve, enclosing $\sigma$)




$I_C$ is exact invariant.

$$I = \oint p\, dq$$

so $\quad I = I_c \Big|_{E,\ \lambda \text{ constant}}$

is <u>approximation</u> to Poincare invariant

for $\dot{\lambda}/\lambda < \omega_0$. $\Bigg\{$ Hence <u>adiabatic</u> invariant.

long time scales

Now, adiabatic invariant:

$$I = \oint_{\lambda, E \text{ fixed}} dq\, p/2\pi \qquad \rightarrow \text{ what is it?}$$

<u>so</u> $\quad I = I(E)$

$$= \oint \frac{p\, dq}{2\pi}$$

$$2\pi \frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E} dq = \oint \frac{dq}{\partial H/\partial p} = \tau$$

$$\therefore \boxed{\partial I/\partial E = 1/\omega}$$

$$\boxed{\partial E/\partial I = \omega}$$

[redacted]

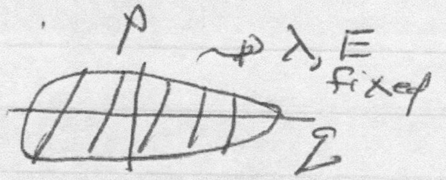
Now, of course:

adiabatic invariant has geometrical significance.

$$I = \oint_{E,\lambda} \frac{p\,dq}{2\pi} = \iint_{E,\lambda} \frac{dp\,dq}{2\pi}$$

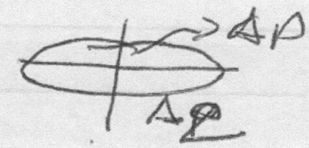
$\therefore$ $I$ corresponds to enclosed area!

i.e.




e.g.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = E$$




$$\Delta p = (2mE)^{1/2}$$
$$\Delta q = (2E/m\omega^2)^{1/2}$$

$$\text{Area} = \pi\,\Delta q\,\Delta p = 2\pi E/\omega$$

$$\boxed{I = E/\omega} \Rightarrow$$

for oscillator, adiabatic invariant is <u>action</u>, $E/\omega$.

$\therefore$ $\dot{\ell}/\ell < \omega_0 \Rightarrow$

$$E \sim \omega \sim \sqrt{g/\ell}$$

Physics 200 A, B — Mechanics

→ Adiabatic Invariants: Review

→ if $H = H(p, q, \lambda(t))$

parametric dependence

with a) periodic motion, for fixed $\lambda$,

b.) $\frac{1}{\lambda}\frac{d\lambda}{dt} < \omega$

rate of change of parameter — motion frequency

then $I_\lambda = \oint_{C_\lambda} p \cdot dq \equiv$ action computed at fixed value of $\lambda$ is adiabatic invariant

→ adiabatic invariant is C.O.M. on time scales $\tau > \omega^{-1}$.

→ adiabatic invariant is intrinsically / implicitly referenced to a given time scale. Same system can manifest multiple adiabatic invariants on different time scales.

→ $I = \oint_C p \cdot dq$ → Poincare-Cartan Invariant → exact COM

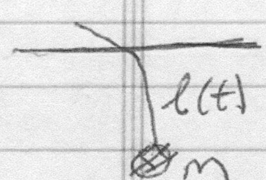
to calculate explicitly, need integrable motion (as in explicit representation of action-angle var.)

but

— $I_\lambda = \oint_{C_\lambda} \underline{p}\cdot d\underline{q}$ is approximation to $I$,

computed for fixed $\lambda$. $\dot{I}_\lambda \cong 0$ for $t >> \omega^{-1}$.

Examples: i) Pendulum — the Prototype




$$\frac{\dot{l}(t)}{l} < \sqrt{g/l}\,.$$

How does $\Theta$ vary with $l$?

$I = E/\omega$, understood $E = \overline{E}$

$\omega = \sqrt{g/l}$

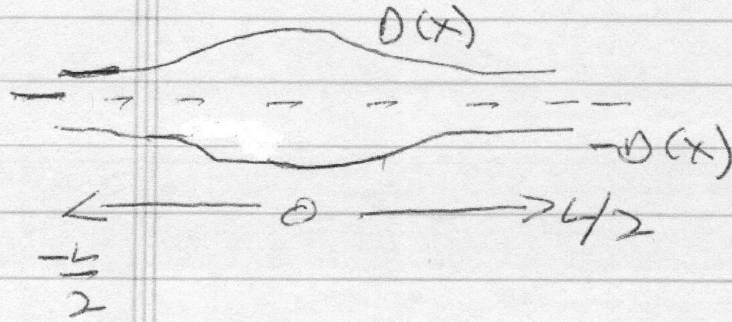
$$\overline{E} = \frac{m}{2} l^2 \overline{\dot{\Theta}^2} + mgl\,\frac{\overline{\Theta^2}}{2}$$

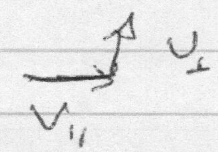
$$= mgl\,\overline{\Theta^2}$$

$$I = m\sqrt{g}\, l^{3/2}\,\overline{\Theta^2}$$

so $\Theta_{rms} \sim l^{-3/4}$ i.e. amplitude decreases as length increases more generally, $\dfrac{\theta_{rms}(t)}{\theta_{rms}(0)} \sim \left(\dfrac{\ell(0)}{\ell(t)}\right)^{3/4}$

## 2.) Mechanical Mirror




$\tau_{b_\perp} \sim \left(v_\perp / 2D\right)^{-1} \rightarrow$ 1 bounce time

$\tau_b \ll L / v_\parallel$

$\Downarrow$

many bounces ($\perp$) in time to sense curvature of D.

now,

$$2\pi I = \int_{-D}^{D} m v_\perp \, dy + \int_{D}^{-D} (-m v_\perp) \, dy$$

$$= 4 m D v_\perp$$

$$\boxed{I = \frac{2}{\pi} D m v_\perp}$$

Adiabatic Invariant

$$E = \frac{1}{2} m \left(v_\perp^2 + v_\parallel^2\right)$$

# More on Adiabatic Invariants

→ for parameter $\lambda(t)$ s/t

$$\left.\dot{\lambda}(t)/\lambda < \omega\right] \rightarrow \text{multiple time scale.}$$

$$\frac{d}{dt}\overline{I} = 0 \qquad \overline{I} = \oint p\,dq$$

$E, \lambda$ fixed

$\overline{I}$ → adiabatic invariant

→ adiabatic invariance ⟺ phase symmetry, along $\oint$.

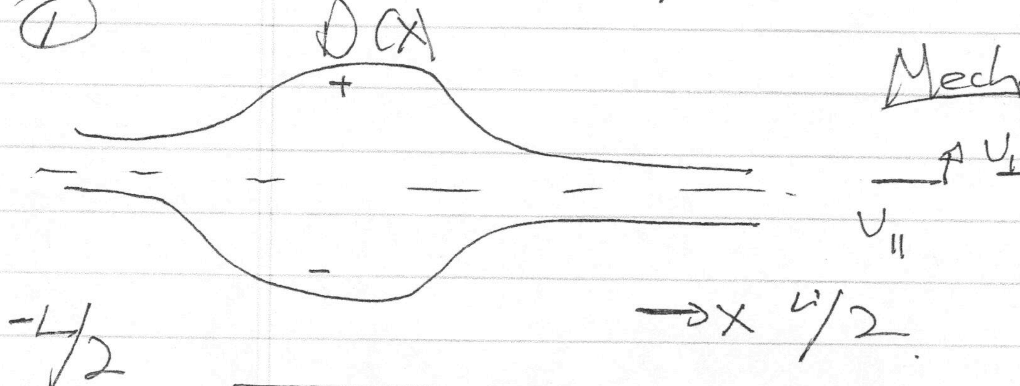
(i.e. can start anywhere in integration).

$$\partial I/\partial E = 1/\omega$$

## Applications of Adiabatic Invariants

Consider 2 related non-trivial (adiabatic invariant - related) systems:

(1) Mechanical Mirror

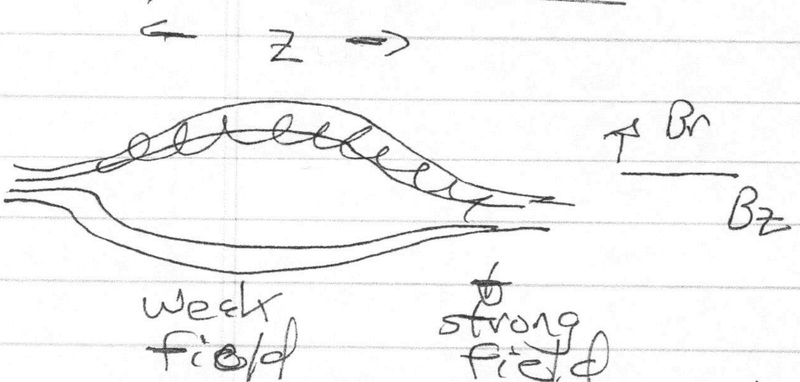



(i.e. curved hard surface)

n.b. $\boxed{D/L \ll 1.}$

cf: video

(2) Magnetic Mirror → basis for mechanical mirror.




$$\nabla \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$

i.e. $\frac{\partial}{\partial z} B_z \neq 0 \Rightarrow \partial_r B_r \neq 0$

for "long, thin" mirror — anisotropy! $\Rightarrow$ $\left\{ \text{long, thin} \Rightarrow \text{slow axial variation} \right.$

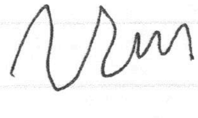
$$B_r \cong -\frac{r}{2} \frac{\partial B_z}{\partial z}\Big|_{r_0}$$

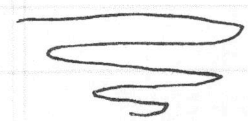
from: $B_r = -\frac{1}{r} \int_0^r dr'\, r' \frac{\partial B_z}{\partial z}$

Consider time scales:

$\rightarrow \tau_{b_\perp} \sim (V_\perp/2D)^{-1} \Rightarrow \perp$ bounce time

$\rightarrow \tau_{b_\parallel} \sim L/V_\parallel \Rightarrow$ parallel bounce time

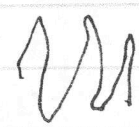
i.e. $\perp$ 

$\parallel$ 

so if consider

$\tau_{b_\perp} < t \Rightarrow$
- many bounces.
- sufficient time to sense curvature of $D$
- can define adiabatic invariant

$$2\pi I = \oint m V_\perp \, dy \rightarrow \underbrace{\oint p_\perp \, dq_\perp}_{\int p_\perp dq_\perp}$$



$$= \int_{-D}^{D} dy \underbrace{m V_\perp}_{\text{forward}} + \int_{D}^{-D} \underbrace{(-m V_\perp)}_{\text{back.}} dy$$

$$= 4 m D V_\perp$$

$$\boxed{I = \frac{2}{\pi} D m V_\perp}$$

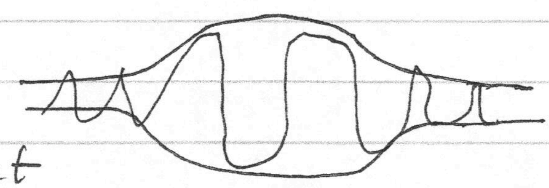
adiabatic invariant on times $t > \tau_{b_\perp}$

i.e. $D v_\perp \sim \text{const}$

$v_\perp$ { large in throat
smaller in
center }

given initial $D(x_0)v_\perp(x_0)$, can determine
$v_\perp(x)$ for all $x$.

Motion ?
{ particle can
reflect from throat }



energy conserved !

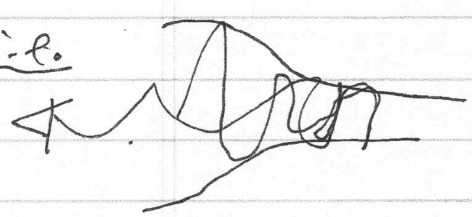
$$E = \frac{1}{2}m\left(v_\perp^2 + v_\parallel^2\right)$$

$$= \frac{1}{2}m\left(v_\parallel^2 + \frac{\pi^2}{4D^2}\frac{I^2}{m^2}\right)$$

$$\Rightarrow v_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x)^2}\frac{I^2}{m^2}$$

so if $I$ s/t $\dfrac{\pi^2 I^2}{4D(x)^2 m^2} > \dfrac{2E}{m}$ $\Rightarrow$ particle reflected in mirror throat.

i.e.



$$I = \frac{2}{\pi} D(x_0)\, m v_{\perp 0}$$

frequently written as:

$$I = \frac{2}{\pi} D(0)\, m v_\perp(0)$$

$x_0 \Leftrightarrow$ center.

 [redacted]

$$\frac{\pi^2 I^2}{4 D(x)^2 M^2} > \frac{2E}{m}$$

i.c.

$$\Rightarrow \left(\frac{D(x_0)}{D(x)}\right)^2 v_\perp^2(x_0) > \frac{2E}{m}.$$

i.c.

for $x < L$ $\Rightarrow$ particle will bounce!

As $E = \frac{1}{2} m \left(v_{\perp 0}^2 + v_{\parallel 0}^2\right)$;

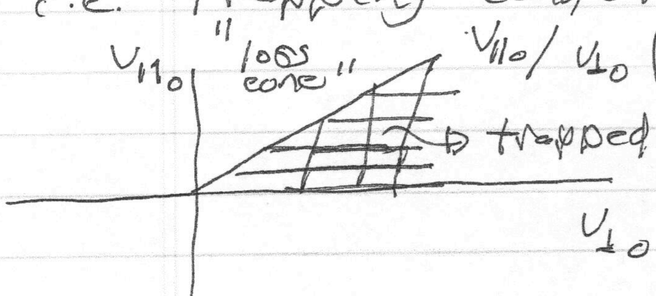
$$\Rightarrow \boxed{\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \left(\frac{D(0)}{D(L)}\right)^2 - 1}$$

"mirror ratio"

i.e. optimal ratio

$$R_m \equiv \frac{D(x_0)^2}{D(x)^2} \to \frac{D(0)^2}{D(L)^2}$$

i.e. trapping condition




(for $x_0 \sim 0$, $x \sim L$)

$$R_m \equiv \frac{D(0)^2}{D(L)^2}$$

basic description of mirror confinement

Now, can determine reflection point

simply by:

$$v_{\parallel}^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x_R)^2}\frac{I^2}{m^2} = 0$$

defines

$x_R \leq L/2$

then: can envision longer times:

$$t > \tau_{b_\parallel} \gg \tau_{b_\perp}$$

$$\tau_{b_\parallel} = \oint \frac{dx}{|v_\parallel|}$$

parallel bounce time, for trapped particles

so

can have "$2^{nd}$" adiabatic invariant on time scale $t > \tau_{b_\parallel} > \tau_{b_\perp}$

$$J_\parallel = \oint dx\, p_\parallel$$

"bounce invariant"

$2^{nd}$.

$J_\perp \Rightarrow$ first adiabatic inv.

$\rightarrow \perp$ bounce.

i.e.

$$T_\perp < \tau < T_\parallel < \tau'$$

$T_\perp$: $\perp$ bounce time

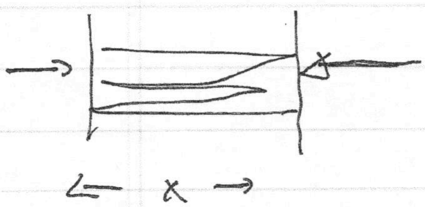
$\tau$: time of $I = J_\perp$ invariant

$T_\parallel$: parallel bounce time

$\tau'$: time of $J_\parallel$ invariant

N.B.: Can expect 1 adiabatic invariant per closed cyclic orbit (n.b. cyclic orbit in action-angle sense).

For application of $J_\parallel$: [Adiabatic compression]



if push slowly:

$$J_\parallel = \oint p_\parallel \, dx = \text{const}$$

$$J_\parallel = \int_{-L}^{L} p_\parallel \, dx + \int_{L}^{-L} -p_\parallel \, dx$$

$$= p_\parallel (2L) - -p_\parallel (2L) = 4L p_\parallel$$

$$\delta J_\parallel = 0 \Rightarrow \delta(p_\parallel L) = 0$$

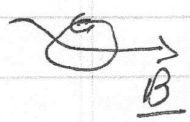
$$\Rightarrow \delta p_\parallel = -\delta L$$

(2) Magnetic Mirror

→ scheme is the same, with magnetic field variation as agent of confinement

→ now, for particle in magnetic field

$$\underline{p} \to \underline{p} - \frac{e}{c}\underline{A} = p_{can}$$



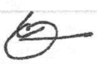

consider cyclotron orbit in plane ⊥ to field

$$\oint_{\perp \text{ plane}} p \, dq = \oint_{cycl.} p_\perp \cdot dq_\perp$$

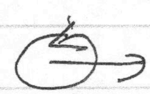
⇒ integrated along Larmor orbit.

$$= \int_C p_{\perp\,can} \cdot dq_\perp - \frac{e}{c}\int_C A_\perp \cdot dq_\perp$$

$$= \int_C m v_\perp \cdot dq_\perp - \frac{e}{c}\int_C A_\perp \cdot dq_\perp$$



Larmor disk



$$= m v_\perp (\rho_L 2\pi) - \frac{e}{c}\pi \rho_L^2 B$$

$\underline{B} = \underline{\nabla} \times \underline{A}$

↓ $2\pi r$ with $r$ = radius of Larmor disk

↳ flux thru Larmor disk.

so

$$\oint_{\perp} p\,dq = m v_\perp \frac{v_\perp}{\frac{eB}{mc}} 2\pi - \frac{e}{c} \pi B \frac{v_\perp^2}{\frac{e^2B^2}{m^2c^2}}$$

$$= \frac{m v_\perp^2}{2B} \left( 4\pi \frac{Mc}{|e|} \right) - \frac{m v_\perp^2}{2B} \left( 2\pi \frac{Mc}{|e|} \right)$$

$$= \frac{m v_\perp^2}{2B} \left( \frac{4\pi Mc}{|e|} \right)$$

irrelevant const.

so $\oint_{\perp} p\,dq = \frac{m v_\perp^2}{2B}$

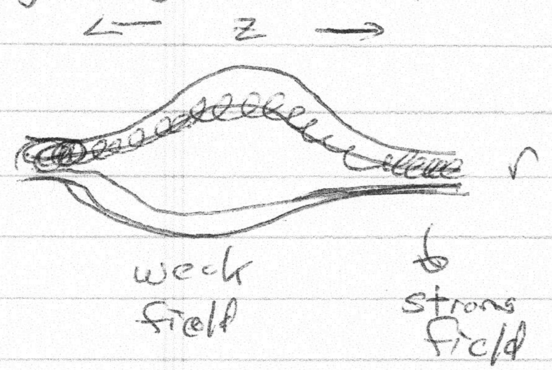
magnetic moment!

Physically: – Magnetic moment corresponds to action computed for 1 cyclotron orbit

– adiabatic invariant on $t > T_{cycl}$, else approx. of closure of cyclotron orbit is meaningless.

[redacted]

3.) Magnetic Mirror — basis for mechanical mirror




$$\underline{\nabla} \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$

$$\neq 0$$

Now, consider rate of change of $\perp$ Energy

$$\frac{d}{dt}\left(\frac{m v_\perp^2}{2}\right) = q\, \underline{E}_\perp \cdot \underline{v}_\perp$$

avg over 1 cyclotron orbit $\Rightarrow$

$$\delta\left(\frac{m v_\perp^2}{2}\right) = \int_{\Omega^{-1}} dt\, q\, \underline{E}_\perp \cdot \underline{v}_\perp$$

$\underline{v}\, dt = l$

change in energy in 1 cyclotron orbit

$$= \oint_{\rho \text{ gyro circle}} d\underline{l} \cdot \underline{E}_\perp q = q \oint \underline{E} \cdot d\underline{l}$$

$\rho \rightarrow$ gyro-radius

$$= \int d\underline{a}\; q \cdot \underline{\nabla} \times \underline{E}$$

via Faraday.

$$= -\int_{\text{gyrodisk}} d\underline{a} \cdot \left(\frac{q}{c} \frac{\partial \underline{B}}{\partial t}\right)$$

$$\cong -\pi \rho^2 \frac{q}{c} \frac{\partial B}{\partial t}$$

$$\rho^2 = v_\perp^2/\Omega^2$$

$\Rightarrow$

$$\delta\left(\frac{m v_\perp^2}{2}\right) \cong -\pi \frac{q}{c} \frac{v_\perp^2}{\frac{q^2 B^2}{m^2 c^2}} \frac{\partial B}{\partial t}$$

$$= -\frac{m v_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but $$\underbrace{\delta B}_{\text{change in 1 cyclotron period}} = \underbrace{\frac{2\pi}{\Omega}}_{\tau_c} \frac{\partial B}{\partial t}$$

i.e.

$$\delta\left(\frac{m v_\perp^2}{2}\right) = -\frac{m v_\perp^2}{2} \frac{1}{B} \delta B$$

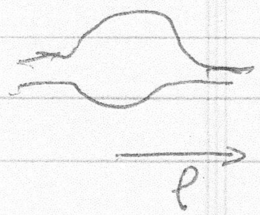
$\Rightarrow$ $$\boxed{\delta\left(\frac{m v_\perp^2}{2B}\right) = 0}$$ $\Rightarrow$ adiabatic time variation in $B$ $\Rightarrow$ heating

so $$\boxed{\mu = m v_\perp^2/2B}$$ $\rightarrow$ magnetic moment adiabatic invariant on $t \gg \Omega^{-1}$

Now, for mirroring:

$$\frac{1}{\cancel{2}}\cancel{m}\left(V_\parallel^2 + V_\perp^2\right) = \frac{1}{\cancel{2}}\cancel{m}\left(V_{\parallel 0}^2 + V_{\perp 0}^2\right)$$

$$\frac{\cancel{m}\, V_\perp^2(0)}{\cancel{2}B(0)} = \frac{\cancel{m}\, V_\perp^2(\ell)}{\cancel{2}B(\ell)}$$




$$V_\parallel^2(0) + V_\perp^2(0) = V_\parallel^2 + \frac{B(\ell)}{B(0)} V_\perp^2(0)$$

$$V_\perp^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) = V_\parallel^2(\ell) - V_\parallel^2(0)$$

for confinement: $V_\parallel^2(\ell) = 0 \Rightarrow$

so $$\boxed{\frac{V_\parallel^2(0)}{V_\perp^2(0)} < \frac{B(\ell)}{B(0)} - 1}$$

mirror ratio

obvious anology to:

$$\frac{V_{\parallel 0}^2}{V_{\perp 0}^2} < \frac{D(x_0)^2}{D(x_R)^2} - 1$$

$B(\ell) \Leftrightarrow 1/D(x_R) \rightarrow$ strong $B \rightarrow$ frequent gyration frequent bouncing

$B(0) \Leftrightarrow 1/D(x_0) \rightarrow$ weak $B \rightarrow$ less frequent bouncing, gyration.

Similarly, can define bounce invariant:

$$J_{\parallel} = \oint dl \left[2m\left(E - \mu B(l)\right)\right]^{1/2}$$

longitudinal action

i.e. $$v_{\parallel}^2(l) = v_{\parallel}^2(0) + v_{\perp}^2(0) - \mu B(l)$$

etc.

squeeze $\rightarrow$ energy gain

N.B.: Treatment of adiabatic invariants given here corresponds to lowest order p.t. in

$$\underbrace{\frac{1}{\lambda}\frac{d\lambda}{dt}\Big/\omega}_{\text{"}O(\epsilon)\text{" here.}} < 1$$

Note: Can also define 'mirror force',

$$F = \frac{q}{c}\, \underline{v} \times B$$

$v_r \quad v_\theta \quad v_z$

$B_r \quad B_\theta \quad B_z$

$$F_z = \frac{q}{c}\left(\cancel{v_r B_\theta} - v_\theta B_r\right)$$

$$\cong \frac{q}{c}\frac{v_\theta}{2}\, r\frac{\partial B_z}{\partial z}$$

$v_\theta \rightarrow v_\perp$

$r \rightarrow \rho_L$

[redacted]

$$F_z \cong \frac{q}{c} \frac{v_\theta r}{2} \frac{\partial B_z}{\partial z}$$

$$= \pm \frac{m v_\perp^2}{2B} \frac{\partial B}{\partial z} = \mp \mu \frac{\partial B}{\partial z}$$

{ depends on location in trajectory